14th International Conference on Information Fusion
Chicago, Illinois, USA, July 5-8, 2011

Nonlinear filtering using measurements affected by stochastic, set-theoretic and association uncertainty

B. Ristic
ISR Division
DSTO
Melbourne
Australia
branko.ristic@dsto.defence.gov.au

A. Gning
School of Comp.& Comm.
Lancaster University
Lancaster
United Kingdom
e.gning@lancaster.ac.uk

L. Mihaylova
School of Comp.& Comm.
Lancaster University
Lancaster
United Kingdom
mila.mihaylova@lancaster.ac.uk

***Abstract*—The problem is sequential Bayesian detection and estimation of nonlinear dynamic stochastic systems using measurements affected by three sources of uncertainty: stochastic, set-theoretic and data association uncertainty. Following Mahler's framework for information fusion, the paper develops the optimal Bayes filter for this problem in the form of the Bernoulli filter for interval measurements, implemented as a particle filter. The numerical results demonstrate the filter performance: it detects the presence of targets reliably, and using a sufficient number of particles, the support of its posterior spatial PDF is guaranteed to include the true target state.**

***Index Terms*—Sequential Bayesian estimation, particle filters, random sets, interval measurements, Bernoulli filter**

I. INTRODUCTION

The problem is sequential Bayesian detection and estimation of dynamic stochastic system using measurements affected by three sources of uncertainty: stochastic, set-theoretic and data association uncertainty. The approach taken in this paper is based on random set theory following Mahler's framework for information fusion [1]. The principle goal is to sequentially estimate the posterior probability of existence and the posterior distribution of a hidden state vector of a dynamic system (target) using all measurements available up to the current time.

Traditionally, the measurements used for nonlinear filtering are points in the measurement space, typically affected by additive measurement noise of a known probability density function (PDF) [2]. The traditional measurements express uncertainty only due to randomness, often referred to as statistical or stochastic uncertainty. In many practical applications, however, the traditional model of measurements is not adequate. In sensor networks, for example, in order to reduce the communication bandwidth, the measurements are quantized to only a few bits. Such measurements represent intervals rather than point values. The intervals express a type of uncertainty which is referred to as the set-theoretic uncertainty [3], vagueness [4] or imprecision [5]. The importance and distinctness of this type of uncertainty have been well recognized by the researchers in expert systems [6]. In the context of Bayes filtering, set-theoretic uncertainty is convenient for modeling bounded errors with unknown distributions and unknown measurement biases. The two types of uncertainties, the set-theoretic and stochastic, can be treated in combination using various modern formalisms, such as: the set of densities [7], imprecise probabilities [8] or random sets [1]. In this paper we will adopt the random set formalism for the combined treatment of imprecision and randomness.

Often, however, the third source of uncertainty in the measurements is present. Due to the imperfections of the detection process, sensors typically operate with probability of detection less than one and, in addition, report measurements which are false [9]. This translates into *data association* uncertainty, that is the uncertainty as to which (if any) of the received measurements is due to the target.

Following Mahler's framework for information fusion [1], the optimal Bayes filter for the described problem of nonlinear filtering using measurements affected by stochastic, set-theoretic and association uncertainty is the Bernoulli filter for unambiguously generated ambiguous (UGA) measurements. The paper develops a particle filter implementation of this filter and studies its performance by numerical simulations.

The rest of the paper is organised as follows. The formal description of the problem is given in Sec.II. The Bernoulli filter for measurements affected by stochastic, set-theoretic and association uncertainty is specified in Sec.III. A particle filter implementation is presented in Sec.IV. The filter performance assessment criteria are described in Sec.V, with numerical studies presented Sec.VI. Finally the conclusions are drawn in Sec.VII.

II. PROBLEM DESCRIPTION

The state vector of the dynamic system (target) at time t_k (discrete-time index k) is denoted by $\mathbf{x}_k$; it takes values from the state space $\mathcal{X} \subseteq \mathbb{R}^{n_x}$. The target, however, may or may not be present in the surveillance region at a particular time t_k. We therefore model the object state at discrete-time k by a finite set $\mathbf{X}_k$ which can be either empty or a singleton. Mahler's *finite set statistics* (FISST) provides practical tools for mathematical manipulations of finite-set random variables, including the concept of probability density function and its integral [1].

A convenient model of target state at time k is the Bernoulli random finite set (RFS) on $\mathcal{X}$. A Bernoulli RFS with parameters q and $s(\mathbf{x})$ has probability $1-q$ of being empty and probability q of being a singleton whose only element is distributed according to the PDF $s(\mathbf{x})$ defined on $\mathcal{X}$. The FISST probability density of a Bernoulli RFS $\mathbf{X}$ is defined as

$$f(\mathbf{X}) = \begin{cases} 1-q, & \text{if } \mathbf{X} = \emptyset \\ q \cdot s(\mathbf{x}), & \text{if } \mathbf{X} = \{\mathbf{x}\} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

The objective of Bayes filtering is to sequentially estimate $\mathbf{X}_k$ from measurements collected up to time k. Suppose the measurement set at time k is denoted by $\boldsymbol{\Upsilon}_k$. Then formally the goal is to estimate sequentially the posterior PDF $f_{k|k}(\mathbf{X}|\boldsymbol{\Upsilon}_{1:k})$ of a Bernoulli random finite process, where $\boldsymbol{\Upsilon}_{1:k} = (\boldsymbol{\Upsilon}_1, \ldots, \boldsymbol{\Upsilon}_k)$ denotes the accumulation of measurement sets up to time k. The estimation is based on prior knowledge of two models, the *target dynamic model* and the *measurement model*.

A. Target dynamic model

Target dynamic model is defined by the probability density $\phi_{k+1|k}(\mathbf{X}|\mathbf{X}')$ associated with moving from state $\mathbf{X}'$ at time k to $\mathbf{X}$ at time $k+1$. Since both $\mathbf{X}'$ and $\mathbf{X}$ are Bernoulli RFSs, $\phi_{k+1|k}(\mathbf{X}|\mathbf{X}')$ can be defined as:

$$\phi_{k+1|k}(\mathbf{X}|\mathbf{X}') = \begin{cases} 1-p_B, & \text{if } \mathbf{X}' = \emptyset, \mathbf{X} = \emptyset \\ p_B \cdot b_{k+1|k}(\mathbf{x}), & \text{if } \mathbf{X}' = \emptyset, \mathbf{X} = \{\mathbf{x}\} \\ 1-p_S(\mathbf{x}'), & \text{if } \mathbf{X}' = \{\mathbf{x}'\}, \mathbf{X} = \emptyset \\ p_S(\mathbf{x}') \cdot \pi_{k+1|k}(\mathbf{x}|\mathbf{x}'), & \text{if } \mathbf{X}' = \{\mathbf{x}'\}, \mathbf{X} = \{\mathbf{x}\} \end{cases} \tag{2}$$

where

- $p_B \stackrel{\text{abbr}}{=} p_{B,k+1|k}$ is the probability of target *birth* during the time interval from k to $k+1$;
- $b_{k+1|k}(\mathbf{x})$ is the spatial distribution of target birth during the time interval from k to $k+1$;
- $p_S(\mathbf{x}') \stackrel{\text{abbr}}{=} p_{S,k+1|k}(\mathbf{x}')$ is the probability that a target with state $\mathbf{x}'$ at time k will survive until time $k+1$;
- $\pi_{k+1|k}(\mathbf{x}|\mathbf{x}')$ is the target transition density from time k to $k+1$.

B. Measurement model

In general, target detection is imperfect: a target may not be detected at scan k, whereas a set of non-existent objects may be detected and reported (false detections or clutter). Let the measurement space be denoted $\mathcal{Z} \subseteq \mathbb{R}^{n_z}$. If target exists, i.e. $\mathbf{X}_k = \{\mathbf{x}\}$, and has been detected, the conventional point measurement $\mathbf{z} \in \mathcal{Z}$ is related to the target state via a nonlinear equation:

$$\mathbf{z} = h_k(\mathbf{x}) + \mathbf{v}, \tag{3}$$

where function h_k is a known deterministic mapping from the state space $\mathcal{X}$ to the measurement space $\mathcal{Z}$, while $\mathbf{v}$ is a measurement noise vector characterised by PDF $p_{\mathbf{v}}$.

In this paper we assume that if target exists ($\mathbf{x} \in \mathbf{X}_k$) and is detected, the sensor does not report the conventional measurement $\mathbf{z} \in \mathcal{Z}$; instead, it reports a closed interval $[\mathbf{z}] \subset \mathcal{Z}$ such that the target originated measurement (3) satisfies $Pr\{h_k(\mathbf{x}) \in [\mathbf{z}]\} = 1$. Let us denote by $\mathcal{IZ}$ the set of all such closed intervals. Note that interval measurements $[\mathbf{z}]$ represent a special case of what Mahler [1] refers to as unambiguously generated ambiguous (UGA) measurements. More general instances of UGA measurements include mixtures of fuzzy membership functions, referred to as fuzzy Dempster-Shafer evidence.

Due to imperfect detection process, $m_k \geq 0$ interval measurements $[\mathbf{z}]_{k,1}, \ldots, [\mathbf{z}]_{k,m_k}$ are collected at time k. The measurements can be represented by a finite set:

$$\boldsymbol{\Upsilon}_k = \{[\mathbf{z}]_{k,1}, \ldots, [\mathbf{z}]_{k,m_k}\} \in \mathcal{F}(\mathcal{IZ}), \tag{4}$$

where $\mathcal{F}(\mathcal{IZ})$ is the space of finite subsets of $\mathcal{IZ}$.

The probability of target detection is assumed to be constant over the state space $\mathcal{X}$, and is denoted by p_D. The false detections are also assumed to be independent of the target state[1]. The number of false detections per scan is modelled by a Poisson distribution with mean λ; the spatial distribution of false detections is modelled by a PDF $c([\mathbf{z}])$.

The measurement set $\boldsymbol{\Upsilon}_k$ is characterised by three sources of uncertainty. Additive noise $\mathbf{v}$ in (3) is the source of stochastic uncertainty. The target originated interval measurement $[\mathbf{z}]$ is non-specific and as such is the source of imprecision. Finally, the existence of false detections and a possible absence of target originated detection is the source of data association uncertainty.

III. BERNOULLI FILTER

The optimal Bayes filter for the problem described above is the Bernoulli filter [1, Sec.14.7], [10] for[2] interval measurements. Let $f_{k|k}(\mathbf{X}|\boldsymbol{\Upsilon}_{1:k})$ denote the posterior PDF of Bernoulli RFS $\mathbf{X}$ at k. The propagation of this posterior over time is carried out in two steps, the *prediction* and *update*. We have seen that $f_{k|k}(\mathbf{X}|\boldsymbol{\Upsilon}_{1:k})$ is completely defined by two posteriors: $q_{k|k} = Pr\{|\mathbf{X}_k| = 1 \mid \boldsymbol{\Upsilon}_k\}$ is[3] the posterior probability of target existence, while $s_{k|k}(\mathbf{x}) = p(\mathbf{x}_k|\boldsymbol{\Upsilon}_{1:k})$ is the posterior spatial PDF of $\mathbf{X}_k = \{\mathbf{x}\}$. The Bernoulli filter requires only these two quantities to be propagated.

A. Equations

Assuming that p_S is state independent, the prediction step equations are given by:

$$q_{k+1|k} = p_B \cdot (1-q_{k|k}) + p_S \cdot q_{k|k} \tag{5}$$

$$s_{k+1|k}(\mathbf{x}) = \frac{p_B \cdot (1-q_{k|k}) b_{k+1|k}(\mathbf{x})}{q_{k+1|k}} + \frac{p_S\, q_{k|k} \int \pi_{k+1|k}(\mathbf{x}|\mathbf{x}') \cdot s_{k|k}(\mathbf{x}')\, d\mathbf{x}'}{q_{k+1|k}} \tag{6}$$

[1] The assumptions about state independent p_D and false detections can be easily relaxed, see [1].

[2] The Bernoulli filter for conventional (point) measurements is referred to as JoTT in [1, Sec.14.7]. It represents a generalisation of the IPDA filter [11], which was derived under the linear-Gaussian-Poisson assumption.

[3] $|\mathbf{X}|$ denotes the cardinality of set $\mathbf{X}$.

The predicted birth density $b_{k+1|k}(\mathbf{x})$ in general is unknown and needs to be adaptively designed using the measurement set $\boldsymbol{\Upsilon}_k$ from the previous scan k. This is further discussed in Sec.IV.

Assuming p_D is constant over the state-space $\mathcal{X}$, the update equations of the Bernoulli filter for interval measurements are as follows [1, Sec.14.7]. The probability of existence is updated using the measurement set $\boldsymbol{\Upsilon}_{k+1}$ as:

$$q_{k+1|k+1} = \frac{1-\delta_{k+1}}{1-\delta_{k+1}\cdot q_{k+1|k}} \cdot q_{k+1|k} \tag{7}$$

where

$$\delta_{k+1} = p_D \left(1 - \sum_{[\mathbf{z}]\in\boldsymbol{\Upsilon}_{k+1}} \frac{\int g_{k+1}([\mathbf{z}]|\mathbf{x})\, s_{k+1|k}(\mathbf{x})\, d\mathbf{x}}{\lambda\, c([\mathbf{z}])}\right). \tag{8}$$

Quantity δ_{k+1} can be positive or negative and can be interpreted as $1-\Lambda_{k+1}$, where Λ_{k+1} is the measurement likelihood ratio under the assumptions of target existence and non-existence. Quantity $g_{k+1}([\mathbf{z}]|\mathbf{x})$ in (8) represents the *generalised* likelihood function at $k+1$ for a target originated interval measurement; furthermore λ and $c([\mathbf{z}])$ are already defined clutter parameters. The generalised likelihood will be further discussed in Sec.III-B.

The target spatial PDF is updated as follows:

$$s_{k+1|k+1}(\mathbf{x}) = \frac{1 - p_D + p_D \sum_{[\mathbf{z}]\in\boldsymbol{\Upsilon}_{k+1}} \frac{g_{k+1}([\mathbf{z}]|\mathbf{x})}{\lambda c([\mathbf{z}])}}{1-\delta_{k+1}}\, s_{k+1|k}(\mathbf{x}) \tag{9}$$

In the spacial case where detection process is perfect, i.e. $p_D = 1$ and no false detections, the measurement set becomes a singleton $\boldsymbol{\Upsilon}_{k+1} = \{[\mathbf{z}]\}$, containing only the target originated measurement. Then it is easy to verify that $\lambda c([\mathbf{z}])$ terms cancel out in (7) and (9) and with $p_B = 0$, $p_S = 1$, the Bernoulli filter for interval measurements becomes identical to the Bayes filter for interval measurements [1, p.159]. For the more general case of $p_D(\mathbf{x})$ and $p_S(\mathbf{x})$, the Bernoulli filter equations can be found in [1, Sec.14.7].

B. Generalised likelihood

The update equations (7) and (9) are different from the standard Bernoulli in the sense that the standard measurement likelihood function is replaced by the *generalised* likelihood function. If $[\mathbf{z}] \in \boldsymbol{\Upsilon}_k$ and $\mathbf{x} \in \mathbf{X}_k$, the generalised likelihood is defined as:

$$g_k([\mathbf{z}]|\mathbf{x}) \overset{\text{def}}{=} Pr\{h_k(\mathbf{x}) + \mathbf{v} \in [\mathbf{z}]\} \tag{10}$$

$$= \int_{[\mathbf{z}]} p_{\mathbf{v}}\Big(\mathbf{z} - h_k(\mathbf{x})\Big)\, d\mathbf{z} \tag{11}$$

Suppose that stochastic uncertainty (which is due to measurement noise $\mathbf{v}$) is small. Then we can adopt the following approximation [1, p.101]:

$$p_{\mathbf{v}}(\mathbf{v}) \approx \mathbb{I}_{[\varepsilon]}(\mathbf{v})/|[\varepsilon]|, \tag{12}$$

where $[\varepsilon]$ is the measurement noise support, $|[\varepsilon]| = \int_{[\varepsilon]} d\mathbf{v}$ is its volume, and $\mathbb{I}_{[\varepsilon]}(\mathbf{v})$ is the indicator function, which equals 1 if $\mathbf{v} \in [\varepsilon]$ and zero otherwise. Using approximation (12) in (11) results in:

$$g_k([\mathbf{z}]|\mathbf{x}) \approx \begin{cases} 1, & \text{if } h_k(\mathbf{x}) + [\varepsilon] \subseteq [\mathbf{z}] \\ 0, & \text{if } h_k(\mathbf{x}) + [\varepsilon] \nsubseteq [\mathbf{z}] \\ \leq 1, & \text{otherwise} \end{cases} \tag{13}$$

In the total absence of randomness (when uncertainty in the measurement $[\mathbf{z}]$ is due to imprecision only), $\varepsilon \to 0$ and the generalised likelihood simplifies to

$$g_k([\mathbf{z}]|\mathbf{x}) = \mathbb{I}_{[\mathbf{z}]}\big(h_k(\mathbf{x})\big). \tag{14}$$

Let $\mathcal{N}(\mathbf{y};\boldsymbol{\mu},\mathbf{P})$ denote a Gaussian PDF with mean $\boldsymbol{\mu}$ and covariance $\mathbf{P}$. Its cumulative distribution function is denoted by $\phi(\mathbf{u};\boldsymbol{\mu},\mathbf{P}) = \int_{-\infty}^{\mathbf{u}} \mathcal{N}(\mathbf{y};\boldsymbol{\mu},\mathbf{P})\, d\mathbf{y}$. Now suppose that measurement noise is zero-mean white Gaussian with covariance matrix $\boldsymbol{\Sigma}$, that is $p_{\mathbf{v}}(\mathbf{v}) = \mathcal{N}(\mathbf{v};\mathbf{0},\boldsymbol{\Sigma})$. Finally let the lower and upper bound of interval $[\mathbf{z}]$ be denoted by $\underline{\mathbf{z}}$ and $\overline{\mathbf{z}}$, respectively, that is $[\mathbf{z}] = [\underline{\mathbf{z}},\ \overline{\mathbf{z}}]$. Then according to (11) the generalised likelihood can be expressed as:

$$\begin{aligned} g_k([\mathbf{z}]|\mathbf{x}) &= \int_{\underline{z}}^{\overline{z}} \mathcal{N}(\mathbf{z}; h_k(\mathbf{x}), \boldsymbol{\Sigma})\, d\mathbf{z} && (15) \\ &= \phi(\overline{\mathbf{z}}; h_k(\mathbf{x}), \boldsymbol{\Sigma}) - \phi(\underline{\mathbf{z}}; h_k(\mathbf{x}), \boldsymbol{\Sigma}) \\ &= 1 - \phi(h_k(\mathbf{x}); \overline{\mathbf{z}}, \boldsymbol{\Sigma}) - (1 - \phi(h_k(\mathbf{x}); \underline{\mathbf{z}}, \boldsymbol{\Sigma})) \\ &= \phi(h_k(\mathbf{x}); \underline{\mathbf{z}}, \boldsymbol{\Sigma}) - \phi(h_k(\mathbf{x}); \overline{\mathbf{z}}, \boldsymbol{\Sigma}) && (16) \end{aligned}$$

Note that this is effectively a fuzzy membership function of $h_k(\mathbf{x})$, as opposed to the crisp membership function expressed by the indicator function in (14). Fig.1 illustrates the generalised likelihood function (16) for one-dimensional measurement ($n_z = 1$), with $\underline{z} = 45$, $\overline{z} = 60$ and three values of Σ, that is 4, 1 and 0.0001. For a very small Σ, the fuzzy membership function (16) effectively becomes the indicator function (14).

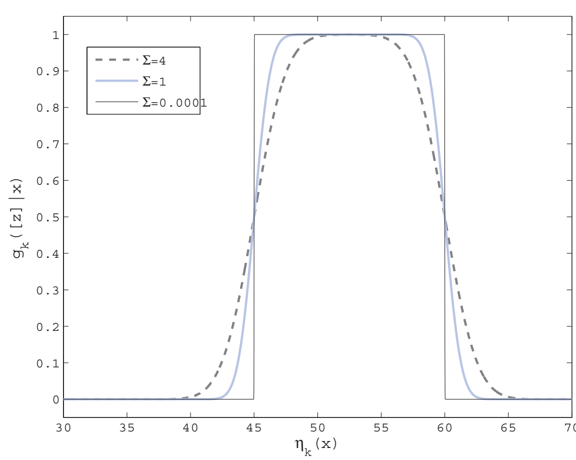


Figure 1. Illustration of the generalised likelihood function (16) for $n_z = 1$: interval measurement $z = [45, 60]$ affected by additive zero-mean Gaussian measurement noise with variance $\boldsymbol{\Sigma}$.

a) Remarks.: The proposed Bernoulli filter is the optimal Bayes filter for the considered problem. Moreover, it is directly applicable to multi-sensor tracking and fusion (unlike for example the PHD filter [12]). While the Bernoulli filter is a single-target tracking algorithm, it can be used in multi-target applications by treating each target separately and by including an appropriate data association technique when targets become close to each other.

IV. Particle filter implementation

In general there is no analytic solution for the Bernoulli filter. Particle filters have become a popular class of numerical methods for implementation of Bayes filters [13], [14], both in the context of single and multiple targets [1]. When this method is applied to the Bernoulli filter, the resulting Bernoulli-particle filter approximates the spatial PDF[4] $s_{k|k}(\mathbf{x})$ by a set of N weighted random samples or particles $\{w_k^i, \mathbf{x}_k^i\}_{i=1}^N$, where $\mathbf{x}_k^i$ is the state of particle i and w_k^i is its corresponding normalised weight, such that $\sum_{i=1}^N w_k^i = 1$. The approximation of $s_{k|k}(\mathbf{x})$ can be written as

$$s_{k|k}(\mathbf{x}) \approx \sum_{i=1}^{N} w_k^i \, \delta_{\mathbf{x}_k^i}(\mathbf{x}) \qquad (17)$$

where $\delta_{\mathbf{a}}(\mathbf{x})$ is the Dirac delta function concentrated at $\mathbf{a}$. For a suitably chosen importance density, the sum in (17) converges to $s_{k|k}(\mathbf{x})$ as $N \to \infty$ [15].

Staring from the posterior Bernoulli density at scan k, represented by $q_{k|k}$ and a set of weighted particles $\{w_k^i, \mathbf{x}_k^i\}_{i=1}^N$, a cycle of the Bernoulli particle filter for interval measurements is summarised in Algorithm 1. The implementation is based on the SIR particle filter, meaning that the transitional density $\pi_{k+1|k}(\mathbf{x}|\mathbf{x}')$ acts as the importance density and that resampling is carried out at every cycle [14].

A. Prediction step

The implementation of prediction equation (6) requires to draw samples from two densities. The predicted birth density $b_{k+1|k}(\mathbf{x})$ is implemented as:

$$b_{k+1|k}(\mathbf{x}) = \int \pi_{k+1|k}(\mathbf{x}|\mathbf{x}') \, b_k(\mathbf{x}') \, d\mathbf{x}' \qquad (18)$$

where $b_k(\mathbf{x})$ is the birth density at the previous time step k. If the target can appear anywhere in the state space $\mathcal{X}$, an obvious choice for $b_k(\mathbf{x})$ is the uniform density over $\mathcal{X}$. This, however, would be very inefficient as it would require a massive number of particles. Instead we design $b_k(\mathbf{x})$ adaptively, using the measurement set from the previous scan k, $\boldsymbol{\Upsilon}_k$, i.e.

$$b_k(\mathbf{x}) \approx \frac{1}{|\boldsymbol{\Upsilon}_k|} \sum_{[\mathbf{z}] \in \boldsymbol{\Upsilon}_k} \beta(\mathbf{x}|[\mathbf{z}]). \qquad (19)$$

Each density $\beta(\mathbf{x}|[\mathbf{z}])$ in the mixture (19) is constructed to be compatible with interval measurement $[\mathbf{z}] \in \boldsymbol{\Upsilon}_k$. By this we mean that $\{\mathbf{x}_{b,k}^i\}_{i=1}^{N_b}$ is a random sample from $\beta(\mathbf{x}|[\mathbf{z}])$ if $[\mathbf{z}]$ can be considered as a draw from $g([\mathbf{z}]|\mathbf{x}_{b,k}^i)$. The number of newborn particles depends on the cardinality of $\boldsymbol{\Upsilon}_k$, that is $N_b = |\boldsymbol{\Upsilon}_k| \cdot n_0$, where n_0 is a design parameter. The weights associated with newborn particles are made equal, i.e. $w_{b,k}^i = 1/N_b$ for $i = 1, \ldots, N_b$. Particles are drawn from $b_k(\mathbf{x})$ in Step 4 of Algorithm 1.

Two bags of weighted particles, the "persistent" and the "newborn" particles, approximate the predicted spatial PDF of (6). The summation of the two terms on the right-hand side of (6) is carried out by the union of these two sets of particles (Step 7 in Alg.1). The number of predicted particles is then $N' = N + N_b$. Their respective weights are computed according to (6), see Step 6 in Alg.1.

B. Update step

The update equations of the Bernoulli particle filter are implemented by steps 8-13 of Alg.1. Using approximation $s_{k+1|k}(\mathbf{x}) \approx \sum_{i=1}^{N'} w_{k+1|k}^i \, \delta_{\mathbf{x}_{k+1|k}^i}(\mathbf{x})$, one can approximate δ_{k+1} of (8) as:

$$\delta_{k+1} \approx p_D \left(1 - \sum_{[\mathbf{z}] \in \boldsymbol{\Upsilon}_{k+1}} \frac{\sum_{i=1}^{N'} g_{k+1}([\mathbf{z}]|\mathbf{x}_{k+1|k}^i) \, w_{k+1|k}^i}{\lambda \, c([\mathbf{z}])} \right) \qquad (20)$$

The generalised likelihood $g_{k+1}([\mathbf{z}]|\mathbf{x}_{k+1|k}^i)$ in (20) is computed according to (11) in the general case and according to (16) if the measurement noise is additive Gaussian. The probability of existence is then updated as in (7), while the weights of the particles are updated following (9) as:

$$\tilde{w}_{k+1}^{i*} = \frac{1 - p_D + p_D \sum_{[\mathbf{z}] \in \boldsymbol{\Upsilon}_{k+1}} \frac{g([\mathbf{z}]|\mathbf{x}_{k+1|k}^i)}{\lambda \, c([\mathbf{z}])}}{1 - \delta_{k+1}} \cdot w_{k+1|k}^i \qquad (21)$$

The updated weights are then normalised to obtain $w_{k+1}^{i*} = \tilde{w}_{k+1}^{i*} / \sum_{j=1}^{N'} \tilde{w}_{k+1}^{j*}$. Finally we resample N times from $\{w_{k+1}^{i,*}, \mathbf{x}_{k+1|k}^i\}_{i=1}^{N'}$ to obtain a random sample $\{w_{k+1}^i = \frac{1}{N}, \mathbf{x}_{k+1}^i\}_{i=1}^N$. In order to prevent sample impoverishment, the resampling step can be followed by regularisation [14]. The filter reports the posterior probability of existence $q_{k+1|k+1}$ and the particle approximation of the posterior spatial PDF $s_{k+1|k+1}(\mathbf{x})$. Since the output weights w_{k+1}^i in Step 14 of Alg.1 are equal, strictly speaking it is unnecessary to input/output them.

We point out that the conventional point state estimates, such as the expected or the maximum a posteriori estimates, would be inappropriate for this filter. As a consequence of imprecise measurements (which model bounded errors with unknown measurement biases), the conventional point state estimates would be also biased.

V. Performance assessment

Since the conventional point state estimates are inappropriate, we cannot use the standard filter error performance measures, such as the mean-square error. How then to assess the error performance of the proposed nonlinear filter?

[4]Strictly speaking particle filters approximate integrals, not densities, [13], [14].

Algorithm 1. The steps of the Bernoulli particle filter for interval measurements

1: **Input**: $q_{k|k}$, $\left\{w_k^i, \mathbf{x}_k^i\right\}_{i=1}^N$, $\mathbf{\Upsilon}_k$, $\mathbf{\Upsilon}_{k+1}$;

Prediction

2: Compute $q_{k+1|k}$ using (5)

3: Draw *persistent* particles at $k+1$: $\mathbf{x}_{p,k+1}^i \sim \pi_{k+1|k}(\mathbf{x}|\mathbf{x}_k^i)$ for $i = 1, \ldots, N$

4: Create a weighted set of *newborn* particles $\{w_{b,k}^i, \mathbf{x}_{b,k}^i\}_{i=1}^{N_b}$ at k from birth density $b_k(\mathbf{x})$ defined by (19), with $w_{b,k}^i = 1/N_b$;

5: Draw *newborn* particles at $k+1$: $\mathbf{x}_{b,k+1}^i \sim \pi_{k+1|k}(\mathbf{x}|\mathbf{x}_{b,k}^i)$ for $i = 1, \ldots, N_b$

6: Compute the weights at $k+1$:

$$\begin{aligned} w_{p,k+1}^i &= p_S\, q_{k|k}\, w_k^i / q_{k+1|k}; \qquad \text{for } i = 1, \ldots, N \\ w_{b,k+1}^i &= p_B\, (1 - q_{k|k}) w_{b,k}^i / q_{k+1|k}; \quad \text{for } i = 1, \ldots, N_b \end{aligned}$$

7: Union of weighted particles: $\{w_{k+1|k}^i, \mathbf{x}_{k+1|k}^i\}_{i=1}^{N'} = \{w_{b,k+1}^i, \mathbf{x}_{b,k+1}^i\}_{i=1}^{N_b} \cup \{w_{p,k+1}^i, \mathbf{x}_{p,k+1}^i\}_{i=1}^{N}$, where $N' = N + N_b$;

Update

8: For every particle $\mathbf{x}_{k+1|k}^i$, $i = 1, \ldots, N'$ and every measurement $[\mathbf{z}] \in \mathbf{\Upsilon}_{k+1}$, compute the generalised likelihood $g([\mathbf{z}]|\mathbf{x}_{k+1}^i)$ according to (11);

9: Compute δ_{k+1} according to (20);

10: Compute $q_{k+1|k+1}$ according to (7);

11: Compute unnormalised weights $\tilde{w}_{k+1}^{i*}$ according to (21) for $i = 1, \ldots, N'$;

12: Normalise weights: $w_{k+1}^{i*} = \tilde{w}_{k+1}^{i*} / \sum_{j=1}^{N'} \tilde{w}_{k+1}^{j*}$;

13: Resample N times from $\{w_{k+1}^{i,*}, \mathbf{x}_{k+1|k}^i\}_{i=1}^{N'}$ to obtain equally weighted particles $\{w_{k+1}^i = \frac{1}{N}, \mathbf{x}_{k+1}^i\}_{i=1}^N$

14: **Output**: $q_{k+1|k+1}$, $\left\{w_{k+1}^i, \mathbf{x}_{k+1}^i\right\}_{i=1}^N$

An optimal filter for the problem described in the paper has to satisfy two conditions:

1) The true value of the target state vector $\mathbf{x}_k$ must be contained in the support of the posterior spatial PDF $s_{k|k}(\mathbf{x})$;
2) The support of the posterior spatial PDF $s_{k|k}(\mathbf{x})$ is minimal.

Accordingly we propose two assessment criteria: the first is referred to as *inclusion* and verifies condition 1; the second, referred to as *volume*, measures the spread (volume) of $s_{k|k}(\mathbf{x})$. Note that the failure to satisfy condition 1 indicates filter divergence, which is considered as a *catastrophic* event in target tracking. For the proposed particle Bernoulli filter for interval measurements, which is a numerical approximation of the optimal filter, it will be imperative to satisfy condition 1 and desirable to satisfy condition 2.

Let us introduce a *credible set* [16] $\mathbf{C}_k(\alpha)$ associated with the posterior $s_{k|k}(\mathbf{x}) = p(\mathbf{x}_k|\mathbf{\Upsilon}_{1:k})$. This set is defined implicitly as $\mathbf{C}_k(\alpha) \subseteq \mathcal{X}$ such that:

$$\int_{\mathbf{C}_k(\alpha)} s_{k|k}(\mathbf{x})\, d\mathbf{x} = \alpha \tag{22}$$

where $\alpha \in [0, 1]$ is probability. Credible set at $\alpha \to 1$ represents the support of the posterior spatial PDF $s_{k|k}(\mathbf{x})$. The *inclusion criterion* ρ_k is defined as:

$$\rho_k = \begin{cases} 1, & \text{if } \mathbf{x}_k \in \mathbf{C}_k(1) \\ 0, & \text{otherwise} \end{cases}. \tag{23}$$

The *volume criterion* ν_k measures the spread of particles by computing the trace of the particle set covariance matrix:

$$P_{k|k} = \sum_{i=1}^N (\mathbf{x}_k^i - \overline{\mathbf{x}}_k)(\mathbf{x}_k^i - \overline{\mathbf{x}}_k)^\intercal \tag{24}$$

where $\overline{\mathbf{x}}_k = \frac{1}{N}\sum_{i=1}^N \mathbf{x}_k^i$. The two assessment criteria, ρ_k and ν_k, will be computed for all time steps k characterised by $q_{k|k} > \tau$, where $\tau \in [0, 1]$ is the track reporting threshold. Furthermore, in order to establish the expected performance, ρ_k and ν_k will be averaged over independent Monte Carlo runs.

The implementation of inclusion criterion ρ_k in (23) is somewhat involved and requires further explanation. Recall that only a random sample approximation of $s_{k|k}(\mathbf{x})$ is available, i.e. $s_{k|k}(\mathbf{x})$ is represented by a set of equally weighted particles $\mathbf{x}_k^i$, $i = 1, \ldots, N$. In order to establish the inclusion of the true state vector, i.e. $\mathbf{x}_k \in \mathbf{C}_k(1)$, we will apply the kernel density estimation (KDE) method [17]. The (fixed) KDE method places a kernel function ϕ on every particle $\mathbf{x}_k^i$, $i = 1, \ldots, N$. The result is an approximation of the posterior density $s_{k|k}(\mathbf{x})$:

$$s_{k|k}(\mathbf{x}) \approx \tilde{s}(\mathbf{x}) = \frac{1}{NW^{n_x}} \sum_{i=1}^N \phi\left(\frac{\mathbf{x} - \mathbf{x}_k^i}{W}\right) \tag{25}$$

where $\phi(\mathbf{x})$ is the kernel which satisfies $\phi(\mathbf{x}) \geq 0$ and $\int_{\mathcal{X}} \phi(\mathbf{x})\, d\mathbf{x} = 1$, and W is the kernel width parameter. For convenience we adopt the Gaussian kernel with zero-mean and covariance matrix $\mathbf{P}$:

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{n_x/2}\sqrt{|\mathbf{P}|}} \exp\left\{-\frac{1}{2}\mathbf{x}^\intercal\, \mathbf{P}^{-1}\, \mathbf{x}\right\}. \tag{26}$$

The optimal fixed bandwidth (under the assumption that the underlying PDF is Gaussian) for the Gaussian kernel $\phi(\mathbf{x})$ is [17] $W^* = A \cdot N^{\frac{1}{n_x+4}}$, where $A = [4/(n_x + 2)]^{\frac{1}{n_x+4}}$. The covariance $\mathbf{P}$ needs to be estimated from the particles as in (24). Using the KDE approximation (25) is would be possible to compute the boundary of the credible set $\mathbf{C}_k(1)$.

The computation involved, however, would be prohibitively expensive, and we propose a simpler approximation of ρ_k in (23) as follows:

$$\rho_k = \begin{cases} 1, & \text{if } \tilde{s}(\mathbf{x}_k) \geq \min_{i=1,\ldots,N} \tilde{s}(\mathbf{x}_k^i) \\ 0, & \text{otherwise} \end{cases} \qquad (27)$$

where $\mathbf{x}_k$ is the true target state at k and $\tilde{s}$ was defined in (25). The value of $\min_{i=1,\ldots,N} \tilde{s}(\mathbf{x}_k^i)$ in (27) effectively defines the boundary of the credible set $\mathbf{C}_k$ at $\alpha \to 1$. The boundary itself, however, does not need to be computed.

VI. Numerical examples

A. Simulation setup

Consider the problem of tracking a target in two-dimensional plane using range, range-rate and azimuth measurements. The target state vector is $\mathbf{x} = \begin{bmatrix} x & \dot{x} & y & \dot{y} \end{bmatrix}^\intercal$, where (x, y) and $(\dot{x}, \dot{y})$ are target position and velocity, respectively, in Cartesian coordinates. The target is moving according to the nearly constant velocity motion model with transitional density $\pi_{k+1|k}(\mathbf{x}|\mathbf{x}') = \mathcal{N}(\mathbf{x}; \mathbf{F}\mathbf{x}', \mathbf{Q})$. Here

$$\mathbf{F} = \mathbf{I}_2 \otimes \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{Q} = \mathbf{I}_2 \otimes \begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} \\ \frac{T^2}{2} & T \end{bmatrix} \cdot \varpi \qquad (28)$$

with $\otimes$ being the Kronecker product, $T = t_{k+1} - t_k$ the sampling interval and ϖ the intensity of process noise [18]. The target appears at scan $k = 3$ and disappears at scan $k = 54$. Initially (at $k = 0$) the target is located at $(550, 300)$m and is moving with velocity $(-5, -8.5)$m/s. The sensor is static, located at the origin of the $x - y$ plane. Other values are adopted as $\varpi = 0.05$, $T = 1$s, with the total observation interval of 60s.

The measurement function $h_k(\mathbf{x})$ is defined as:

$$h_k(\mathbf{x}) = \begin{bmatrix} \sqrt{x^2 + y^2} \\ \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2+y^2}} \\ \arctan(y/x) \end{bmatrix} \qquad (29)$$

The measurement noise $\mathbf{v}$ is zero-mean white Gaussian with covariance $\mathbf{\Sigma} = \text{diag}[\sigma_r^2, \ \sigma_{\dot{r}}^2, \ \sigma_\theta^2]$, where $\sigma_r = 2.5$m, $\sigma_{\dot{r}} = 0.01$m/s and $\sigma_\theta = 0.25°$. The sensors provides interval measurements, with interval length $\mathbf{\Delta} = [\Delta r, \ \Delta\dot{r}, \ \Delta\theta]^\intercal$, where $\Delta r = 50$m, $\Delta\dot{r} = 0.2$m/s and $\Delta\theta = 4°$ are the lengths of intervals in range, range-rate and azimuth, respectively.

The sensor has a bias (systematic error) in the sense that the vector $h_k(\mathbf{x}) + \mathbf{v}_k$ is not in the middle of the measurement interval. A measurement at k is defined as:

$$[\mathbf{z}]_k = [h_k(\mathbf{x}) + \mathbf{v}_k - \frac{3}{4}\mathbf{\Delta}, \ h_k(\mathbf{x}) + \mathbf{v}_k + \frac{1}{4}\mathbf{\Delta}] \qquad (30)$$

The filter is ignorant of the bias.

The probability of detection is $p_D = 0.95$, the mean number of clutter detections per scan is $\lambda = 5$. The clutter detection spatial distribution $c([\mathbf{z}])$ is uniformly across the range (mid intervals from 30m to 700m), range-rate (mid intervals from -15 to $+15$m/s) and azimuth (mid intervals from $-\pi/2$ to $\pi/2$rad).

The filtering algorithm knows a priori the following: p_D, clutter statistics λ and $c([\mathbf{z}])$, measurement function $h_k(\mathbf{x})$, covariance $\mathbf{\Sigma}$ and the transitional density $\pi_{k+1|k}(\mathbf{x}|\mathbf{x}')$. The filter is making inference at every k using measurements $\mathbf{\Upsilon}_{1:k}$, and the following parameters: $p_B = 0.01$, $p_S = 0.98$, $n_0 = 6500$ and N. The number of particles N will be varied.

Fig.2.(a) shows the tracking scenario under consideration at $k = 51$, with measurements (30). The green regions represent the measurements, the red asterisk is the true target location, while the gray dots are the particles (number of particles $N = 5000$). Although the particle mean $\hat{\mathbf{x}}_{k|k}$ is a biased estimate, the particles populate the volume of the state space $\mathcal{X}$ where the true value resides. Fig.2.(b) shows the estimate of the probability of target existence $q_{k|k}$ over time. Target presence is established at $k = 5$ with $q_{k|k}$ remaining close to 1.0 after that. Occasionally, when the target detection is missing in the measurement set $\mathbf{\Upsilon}_k$, $q_{k|k}$ drops below the value of 1.0.

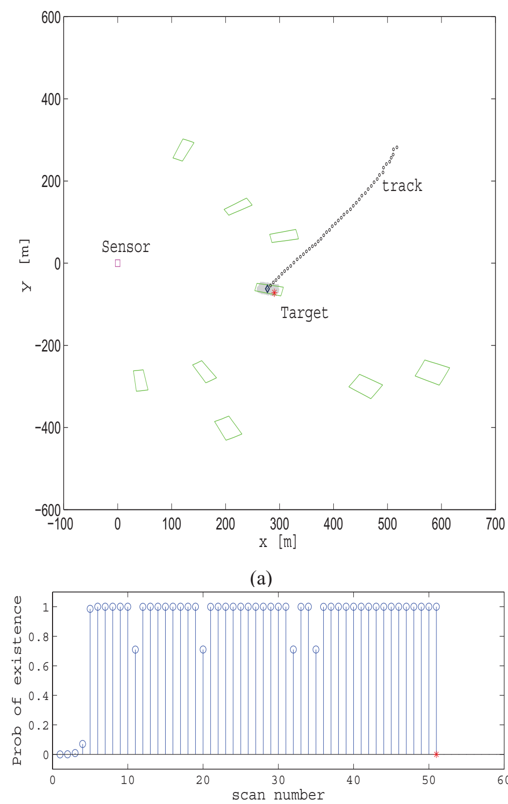
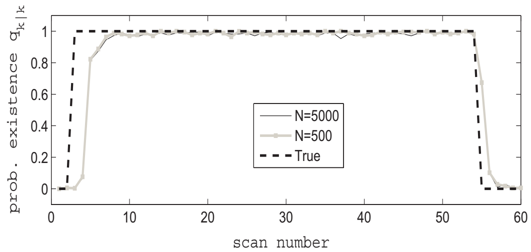


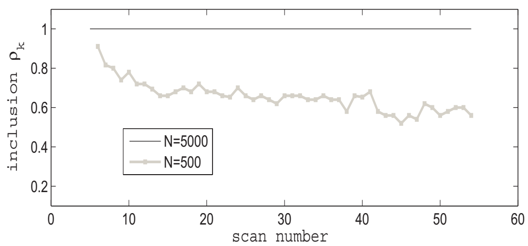
Figure 2. Tracking scenario with results at $k = 51$

B. Monte Carlo runs

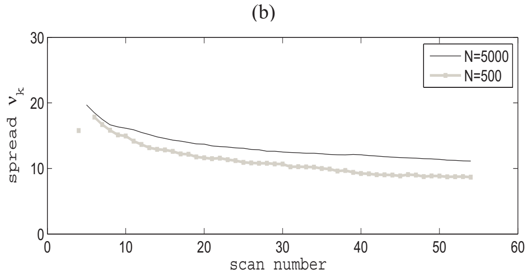
The average performance of the proposed Bernoulli particle filter has been established via Monte Carlo simulations using the scenario and parameters described in Sec. VI-A. Fig. 3 shows: (a) the average probability of target existence $q_{k|k}$; (b) the average *inclusion* criterion ρ_k; (c) the average volume (spread) ν_k, versus the scan number $k = 1, 2, \ldots$. Averaging was carried out over $M = 50$ independent Monte Carlo runs. Two cases for the number of particles N are displayed: $N = 5000$ (solid thin black line) and $N = 500$ (solid thick gray line). The reporting threshold τ was set to 0.5.



(a)



(b)



(c)

Figure 3. Average performance over $M = 50$ Monte Carlo runs using $N = 500$ and $N = 5000$ particles: (a) the probability of existence $q_{k|k}$; (b) the inclusion ρ_k; (c) volume (spread) ν_k.

From Fig. 3.(a) one can observe that the probability of existence is reliable for both values of N. Fig.3.(b) is the key result of this paper: it reveals that by using a sufficient number of particles (in this case $N = 5000$), the average inclusion ρ_k equals 1 for all $k = 1, 2, \ldots$. This means that the true value of the target state $\mathbf{x}_k$ is guaranteed to be contained in the support of the particle representation of $s_{k|k}(\mathbf{x})$. Using smaller number of particles (in this case $N = 500$), the inclusion drops to around 0.6, meaning that in 40% of the runs the filter is diverging. Finally Fig.3.(c) shows that the spread of particles (the volume of the particle representation of $s_{k|k}(\mathbf{x})$). For $N = 500$ this spread is somewhat smaller than for $N = 5000$ case, but as we know from Fig.3.(b), the price of this reduction is filter divergence.

VII. Summary

The paper formulated the optimal Bayesian nonlinear filter for measurements affected by three sources of uncertainty: stochastic, set-theoretic and data association uncertainty. This optimal filter is implemented by a Monte Carlo approximation method, resulting in a particle filter which performs remarkably well when using a sufficient number of particles: the presence of a target is reliably detected, while the true target state is guaranteed to be contained in the support of the particle representation of the spatial density function.

The numerical results have also revealed that the proposed particle filter implementation requires a substantial number of particles in order to achieve a satisfactory performance. This requirement is not unexpected, considering that the support of the spatial density function is large in the presence of three sources of uncertainty in measurements. In the accompanying paper [19], an alternative particle implementation of the Bernoulli filter for measurements affected by three sources of uncertainty is investigated. The basic idea is to replace a particle by a *box particle* [20], which occupies a tiny (but nonzero) fraction of the state-space, rather than a point. The box-particle implementation hence has a potential to significantly reduce the number of particles without a loss in the error performance.

Future work will focus on the development of a multi-Bernoulli filter for multi-target tracking in the presence of stochastic, set-theoretic and data association uncertainty.

Acknowledgements. A. Gning and L. Mihaylova acknowledge the support of the EPSRC project EP/E027253/1 and the [European Community's] Seventh Framework Programme [FP7/2007-2013] under grant agreement No 238710 (Monte Carlo based Innovative Management and Processing for an Unrivalled Leap in Sensor Exploitation).

References

[1] R. Mahler, *Statistical Multisource Multitarget Information Fusion*. Artech House, 2007.
[2] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*. Academic Press, 1970.
[3] P. L. Combettes, "Foundations of set theoretic estimation," *Proc. IEEE*, vol. 81, no. 2, pp. 182–208, 1993.
[4] R. Kruse, E. Schwecke, and J. Heinsohn, *Uncertainty and vagueness in knowledge based systems*. Berlin: Springer-Verlag, 1991.
[5] P. Smets, "Imperfect information: Imprecision and uncertainty," in *Uncertainty Management in Information Systems*, 1996, pp. 225–254.
[6] G. J. Klir and M. J. Wierman, *Uncertainty-based information: Elements of generalized information theory*, 2nd ed. New York: Physica-Verlag, 1999.
[7] V. Klumpp, B. Noack, M. Baum, and U. D. Hanebeck, "Combined set-theoretic and stochastic estimation: A comparison of the SSI and CS filter," in *Proc. 13th Int. Conf. Information Fusion*, Edinburgh, UK, July 2010.
[8] A. Benavoli and A. Antonucci, "Aggregating imprecise probabilistic knowledge," in *Proc. 6th International Symposium on Imprecise Probability*, Durham, UK, 2009.
[9] S. M. Kay, *Fundamentals of Statistical Signal Processing: Detection theory*. Prentice Hall, 1998.
[10] B.-T. Vo, "Random finite sets in multi-object filtering," Ph.D. dissertation, School of EECE, The University of Western Australia, June 2008.
[11] D. Mušicki, R. Evans, and S. Stankovic, "Integrated probabilistic data association," *IEEE Trans. Automatic Control*, vol. 39, no. 6, pp. 1237–1240, June 1994.
[12] R. Mahler, "The multisensor PHD filter II: Errorneous solution via "Poisson magic"," in *Proc. SPIE*, vol. 7336, 2009.

[13] A. Doucet, J. F. G. de Freitas, and N. J. Gordon, Eds., *Sequential Monte Carlo Methods in Practice*. New York: Springer, 2001.
[14] B. Ristic, S. Arulampalam, and N. Gordon, *Beyond the Kalman filter: Particle filters for tracking applications*. Artech House, 2004.
[15] D. Crisan and A. Doucet, "A survey of convergence results on particle filtering methods for practitioners," *IEEE Trans. Signal Processing*, vol. 50, no. 3, pp. 736–746, 2002.
[16] J. O. Berger, *Statistical Decision Theory and Bayesian Analysis*. New York: Springer Series in Statistics, 1985.
[17] B. W. Silverman, *Density estimation for statistical and data analysis*. Chapman and Hall, 1986.
[18] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation*. John Wiley & Sons, 2001.
[19] A. Gning, B. Ristic, and L. Mihaylova, "A box particle filter for stochastic set-theoretic measurements with association uncertainty," in *Proc. 14th Int. Conf. Information Fusion*, Chicago, USA, July 2011, (This volume).
[20] F. Abdallah, A. Gning, and P. Bonnifait, "Box particle filtering for nonlinear state estimation using interval analysis," *Automatica*, vol. 44, pp. 807–815, 2008.